Conditions for the Equivalence of the Chen and Yee FDTD Algorithms

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***Abstract*— A two-dimensional dispersion analysis of the recently proposed Chen Finite-Difference Time-Domain (FDTD) algorithm is presented and its properties evaluated. The dispersion properties of this scheme vary with respect to propagation angle, Courant stability factor, and the definition of numerical characteristic impedance. These dispersion characteristics can be used to identify parameters that make the Chen algorithm equivalent to the Yee FDTD algorithm.**

I. Introduction

Recently, Chen *et al.* [1] proposed a new 3-D Finite-Difference Time-Domain (FDTD) algorithm which was constructed to be equivalent to Johns' Symmetrical Condensed Node Transmission-Line Matrix (SCN-TLM) method [2]. The authors also proposed a 2-D FDTD algorithm equivalent to the 2-D TLM shunt node formulation [3]. The equivalence of Chen's new FDTD algorithms to the classical Yee FDTD algorithm and the various TLM formulations has been the subject of recent debate [4]. Previous dispersion analysis has shown an equivalence between Yee's FDTD algorithm and the shunt node TLM formulation under certain conditions [5]. Any such equivalence depends on the choice of several parameters, including the Courant stability factor and the numerical characteristic impedance. In this note, we identify the specific conditions under which the Chen and Yee FDTD algorithms are equivalent.

An interesting aspect of Chen's 3-D formulation is that it does not reduce to his 2-D algorithm with an invariance in one direction. Specifically, in his 2-D formulation, the numerical characteristic impedance, Z_{num}, is defined such that $Z_{\text{num}} = \sqrt{2}Z_o$, whereas in 3-D it is defined such that $Z_{\text{num}} = Z_o$. ($Z_o = \sqrt{\frac{\mu}{\epsilon}}$ is the intrinsic impedance of free space.) In fact, the definition of Z_{num} is arbitrary, in the sense that any value of $Z_{\text{num}} = \alpha Z_o$ can provide a stable time-marching algorithm. However, the dispersion properties of the method are highly dependent on the choice of α.

In this letter, the dispersion relation of the generalized 2-D Chen FDTD algorithm is given, by leaving α as a free parameter. Two specific cases are studied; the $\alpha = \sqrt{2}$ case, which is the 2-D formulation given in [1], and the case where $\alpha = 1$, which is a formulation where the Chen 3-D FDTD equations are reduced to a set of 2-D FDTD equations by enforcing an invariance in one direction. From the dispersion properties of each algorithm, equivalences with the classical 2-D Yee algorithm and with Johns' TLM formulations will be noted.

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II. Dispersion Relation

Assuming a time harmonic solution to the generalized 2-D Chen FDTD equations [1, eqns (7)–(9) and (14)–(17) with $Z_{\text{num}} = \alpha Z_o$], for an isotropic, homogeneous, linear and lossless medium, the dispersion relation in matrix form is

$$\begin{vmatrix} A & -D & E \\ -D & B & 0 \\ E & 0 & C \end{vmatrix} = 0, \qquad (1)$$

where

$$A = 2j\sin\left(\frac{\omega\Delta t}{2}\right) - \frac{1}{\alpha}(s_x B_x + s_z B_z),$$
$$B = 2j\sin\left(\frac{\omega\Delta t}{2}\right) - \alpha s_z B_z,$$
$$C = 2j\sin\left(\frac{\omega\Delta t}{2}\right) - \alpha s_x B_x,$$
$$D = s_z A_z,$$
$$E = s_x A_x.$$

In (1), the A_x and B_x are defined as

$$A_x = \frac{D_x\Gamma_x - \frac{1}{2}D_x\delta_x^2 e^{-j\omega\Delta t/2}}{\Gamma_x^2 - \frac{1}{4}D_x^2 e^{-j\omega\Delta t}},$$
$$B_x = \frac{\Gamma_x\delta_x^2 - \frac{1}{2}D_x^2 e^{-j\omega\Delta t/2}}{\Gamma_x^2 - \frac{1}{4}D_x^2 e^{-j\omega\Delta t}},$$

The A_z and B_z are defined similarly. Other terms are defined as

$$s_x = c\frac{\Delta t}{\Delta x} \qquad s_z = c\frac{\Delta t}{\Delta z},$$
$$D_x = 2j\sin(k_x\Delta x), \quad \delta_x = 2j\sin(\frac{k_x\Delta x}{2}),$$
$$\Gamma_x = e^{j\omega\Delta t/2} + e^{-j\omega\Delta t/2}\cos(k_x\Delta x).$$

In (1), Δx and Δz are the grid spacings, Δt is the time step, ω is the angular frequency of the propagating wave, k_x and k_z are the numerical wavenumbers in the x and z directions respectively, and $j = \sqrt{-1}$.

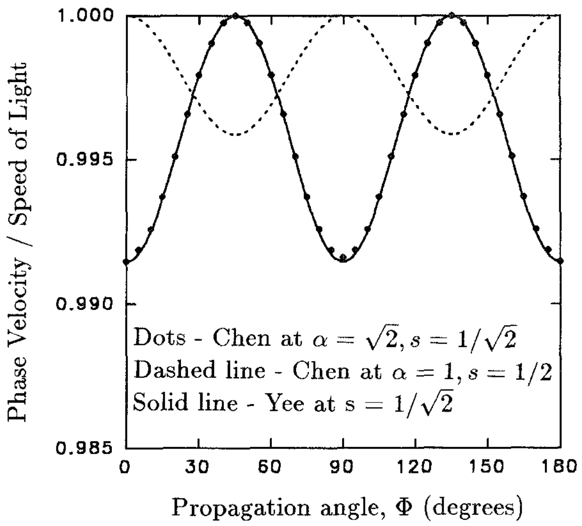

Fig 1 Phase velocity versus propagation angle for the various schemes.

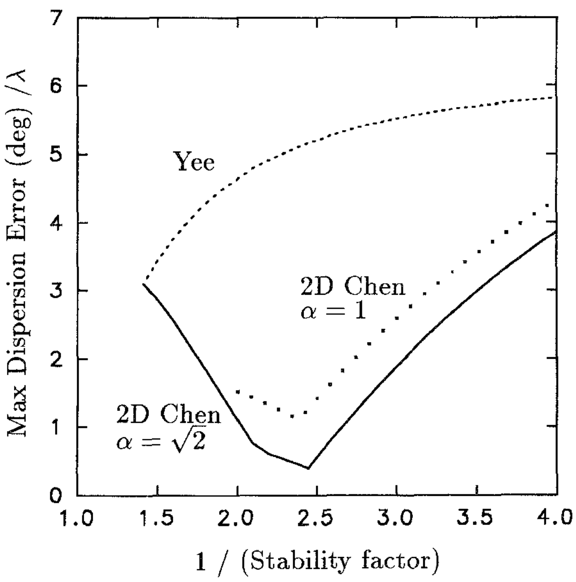

Fig. 2. Maximum dispersion error versus stability factor for the various schemes.

The numerical phase velocity and grid dispersion per wavelength are determined from the numerical wavenumbers k_x and k_z [6].

III. Comparisons

For all comparisons, the spatial grid is uniform ($\delta = \Delta x = \Delta z$), and the spatial grid increment is $\delta = \lambda_o/10$, where λ_o is the free space wavelength. The Courant stability factor is defined as $s = c\Delta t/\delta$.

Fig. 1 shows a plot of the phase velocity, normalized to the speed of light, versus propagation angle, ϕ, for the 2-D Chen algorithm for the two different choices of α. The dots indicate the case when $\alpha = \sqrt{2}$ at its Courant stability limit, $s_{\max} = 1/\sqrt{2}$. Also shown in Fig. 1 is the phase velocity for the Yee algorithm (solid line) at its stability limit of $s_{\max} = 1/\sqrt{2}$. Notice that both algorithms exhibit **identical** phase velocity profiles and have the same Courant stability limit. Thus, the Chen algorithm with $\alpha = \sqrt{2}$ is said to be **identical** to Yee's algorithm at this Courant limit (as noted in [6]). (However, this equivalence is only true at this value of stability factor, s).

The dashed line in Fig. 1 represents the 2-D generalized Chen algorithm evaluated with $\alpha = 1$ at its stability limit, $s_{\max} = 1/2$. The Courant stability limit is not constant; it is a function of the definition of α in the numerical characteristic impedance. Furthermore, note that the dispersion properties for the two choices of α are out of phase by 180°. The case of $\alpha = \sqrt{2}$ has zero dispersion (phase velocity = speed of light) along the coordinate diagonals ($\Phi = 45°, 135°$, etc.) and maximum dispersion (phase velocity farthest from the speed of light) along the coordinate axes ($\Phi = 0°, 90°, 180°$, etc.) at its Courant limit, while the case of $\alpha = 1$ has zero dispersion along the coordinate axes, and maximum dispersion along the coordinate diagonals at its Courant limit.

Fig. 1 also illustrates the equivalence between the Chen algorithms and various TLM formulations. The 2-D Chen algorithm with $\alpha = \sqrt{2}$ exhibits the same angular dispersion dependence as the 2-D shunt node TLM, while the 2-D algorithm with $\alpha = 1$ exhibits the same angular dispersion dependence as the 2-D Symmetrical Condensed Node TLM.

Fig. 2 shows a plot of the maximum dispersion error per wavelength as a function of the reciprocal of the stability factor for the two Chen schemes, $\alpha = 1$ and $\alpha = \sqrt{2}$. Also plotted for comparison is the dispersion curve for the Yee algorithm. From this curve, it is clear that not only does the Chen algorithm with $\alpha = \sqrt{2}$ have a lower Courant stability limit than the case $\alpha = 1$, but that the scheme with $\alpha = \sqrt{2}$ has better dispersion properties than the scheme operated at $\alpha = 1$, when each is operated at the same fixed Courant stability factor. Thus, the choice of $\alpha = \sqrt{2}$ would be preferable in an FDTD code.

IV. Conclusion

The 2-D Chen FDTD scheme with $\alpha = \sqrt{2}$ is equivalent to the 2-D Yee FDTD scheme, but only at $s = 1/\sqrt{2}$. The Chen FDTD algorithm exhibits optimum performance at $s \approx 0.408$.

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